MA114 Summer 2018
Worksheet 9b – Integral Test – 6/25/18

1. Use the Integral Test to determine if the following series converge or diverge.

a) $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ $\quad f(x) = \frac{1}{1+x^2}$ on $[0, \infty)$

• f is definitely positive & cts. Since $1+x^2$ is an increasing function on $[0, \infty)$, $\frac{1}{1+x^2}$ is decreasing.

So we can use the integral test.

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_0^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \arctan(R) - \arctan(0) = \frac{\pi}{2} - 0.$$

Since $\int_0^\infty \frac{1}{1+x^2}\,dx$ converges, $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ also converges by the Integral Test.

b) $\sum_{n=2}^{\infty} \frac{n}{(n^2+2)^{1/2}}$

Whoops. The Integral Test does not apply since $f(x) = \frac{x}{(x^2+2)^{1/2}}$ is increasing.

But $\lim_{n\to\infty} \frac{n \cdot \frac{1}{n}}{\sqrt{n^2+2}\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{2}{n^2}}} = \frac{1}{\sqrt{1+0}} = 1$, so the series diverges

by the Divergence Test

c) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ $f(x)=x^2e^{-x^3}$ is clearly continuous and positive on $[1,\infty)$.

$f'(x)=2xe^{-x^3}-3x^4e^{-x^3}=\underbrace{x}_{>0}\underbrace{e^{-x^3}}_{>0}\underbrace{(2-3x^3)}_{<0}$ for $x>1$. So $f'(x)<0$ for $x\geq 1$.

Thus we can use the Integral Test.

$$\int_1^\infty x^2e^{-x^3}dx=\lim_{R\to\infty}\int_1^R x^2e^{-x^3}dx=\lim_{R\to\infty}\int -\frac{1}{3}e^u du=\lim_{R\to\infty}-\frac{1}{3}e^{-x^3}\Big|_1^R$$

$u=-x^3$
$du=-3x^2dx$

$$=\lim_{R\to\infty}-\frac{1}{3e^{-R^3}}+\frac{1}{3e}$$

$$=\frac{1}{3e}.$$

Since $\int_1^\infty x^2e^{-x^3}dx$ converges, so does $\sum_{n=1}^{\infty} n^2e^{-n^3}$!

d) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

$f(x)=\frac{\ln(x)}{x}$ is continuous and positive once $x>1$ (since $\ln(1)=0$)

$f'(x)=\frac{\frac{1}{x}\cdot x-\ln(x)\cdot 1}{x^2}=\frac{1-\ln(x)}{x^2}<0$ for $x>e$ since $\ln(x)>1$ if $x>e$.

So we can use the integral test:

$u=\ln(x)$

$$\int_1^\infty \frac{\ln(x)}{x}dx=\lim_{M\to\infty}\int_1^M \frac{\ln(x)}{x}dx=\lim_{M\to\infty}\int_0^{\ln M} u\,du=\lim_{M\to\infty}\frac{1}{2}(\ln M)^2=\infty.$$

Since $\int_1^\infty \frac{\ln(x)}{x}dx$ diverges, so does $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$.